

Hamiltonian Decomposition of Lexicographic Product

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Communicated by Vera T. Sós

Received November 16, 1978

In this paper we prove the conjecture of J.-C. Bermond (*Ann. Discrete Math.* 36 (1978), 21–28): If two graphs are decomposable into Hamiltonian cycles, then their lexicographic product is decomposable, too.

1. INTRODUCTION

Let $G_1 = (X_1, E_1)$ and $G_2 = (X_2, E_2)$ be two simple graphs. Their cartesian product is the graph which has the cartesian product $X_1 \times X_2$ as the vertex set, and vertices (x_1, x_2) and (y_1, y_2) are joined if $((x_1, y_1) \in E_1 \text{ and } x_2 = y_2)$ or $(x_1 = y_1 \text{ and } (x_2, y_2) \in E_2)$. The lexicographic product $G_1 \otimes G_2$ has the same vertex-set and the vertices (x_1, x_2) and (y_1, y_2) are joined iff $(x_1, y_1) \in E_1$ or $(x_1 = y_1 \text{ and } (x_2, y_2) \in E_2)$. We denote by S_n the graph consisting of n isolated vertices and by C_n the cycle of length n ($n \geq 3$).

Kotzig has shown that $C_r + C_n$ is decomposable into two (edge-disjoint) Hamiltonian cycles [4]. We will use this result in the case r even n odd. The decomposition of $K_r \otimes S_n (= K_{r \times n})$ was studied in [1, 3]. The fact that $C_r \otimes S_n$ is always decomposable into Hamiltonian cycles was proved in [3, 5].

Laskar proved [5] that $C_r \otimes C_n$ is decomposable into Hamiltonian cycles if r odd and in some additional cases. Bermond [2] conjectured that if G_1 and G_2 are decomposable into Hamiltonian cycles then $G_1 \otimes G_2$ is decomposable, too. To prove this assertion we need some constructions used by Laskar [5]. For the sake of completeness we expound these constructions in the following section.

2. SOME DECOMPOSITIONS

Let $|X_1| = r$, $|X_2| = n$. We denote the elements of $X_1 \times X_2$ by x_{ij} ($i = 0, \dots, r-1$, $j = 0, \dots, n-1$). Let F_{it} be defined by $F_{it} = \{(x_{ij}, x_{i+1, j+i})$;

$j = 0, \dots, n-1$ for $t = 0, \dots, r-1$, $i = 0, \dots, n-1$, where the indices are considered modulo r resp. modulo n .

Clearly $\bigcup_{t=0}^{r-1} \bigcup_{i=0}^{n-1} F_{ti}$ is just the set of the edges of $C_r \otimes S_n$.

$C_r \otimes S_n$ is decomposable into Hamiltonian cycles H_0, \dots, H_{n-1} in the following way:

1. If r and n are odd let $H_i = F_{0,i} \cup F_{1,-i} \cup F_{2,i} \cup \dots \cup F_{r-4,-i} \cup F_{r-3,i} \cup F_{r-2,i} \cup F_{r-1,-2i+1}$ ($i = 0, \dots, n-1$).

Clearly H_i is a Hamiltonian cycle and H_i and $H_{i'}$ have no common edges if $i \neq i'$.

2. If r is even let $H_i = F_{0,i} \cup F_{1,-i} \cup F_{2,i} \cup \dots \cup F_{r-3,-i} \cup F_{r-2,i} \cup F_{r-1,-i+1}$ ($i = 0, \dots, n-1$).

3. If $n = 2$, H_0 and H_1 are given by Fig. 1.

4. If n is even then $C_r \otimes S_n$ is decomposable into two graphs isomorphic to $C_{2r} \otimes S_{n/2}$. This decomposition is given in 3. Indeed let us divide the set V_i into two parts W_i and W'_i ($W_i \cap W'_i = \emptyset$ | $|W_i| = |W'_i| = n/2$) and consider the sets W_i and W'_i as the vertices of $C_r \otimes S_2$. The decomposition of $C_{2r} \otimes S_{n/2}$ is given in 2.

5. If n and r are even then the constructions in 2 and in 4 give us different decompositions. Nevertheless for our purpose we need a third type of factorization into Hamiltonian cycles, too.

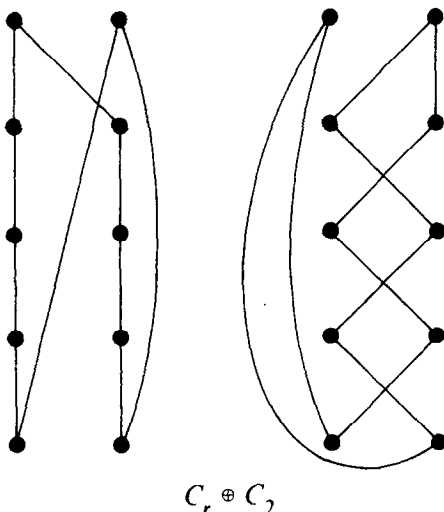


FIGURE 1

Let the Hamiltonian cycles H_1, \dots, H_n be defined by

$$\begin{aligned}
 H_{2i-1} &= \bigcup_{t \text{ even}} ((F_{t,2i-1} \cup F_{t,2i}) \setminus (x_{t,0}, x_{t+1,2i-1})) \\
 &\quad \cup \bigcup_{t \text{ odd}} x_{t,2i-1}, x_{t+1,0}, \\
 H_{2i} &= \bigcup_{t \text{ odd}} ((F_{t,1-2i} \cup F_{t,-2i}) \setminus (x_{t,2i-1}, x_{t+1,0})) \\
 &\quad \cup \bigcup_{t \text{ even}} (x_{t,0}, x_{t+1,2i-1}),
 \end{aligned} \tag{1}$$

where t runs from 0 to $r-1$ and $i = 1, \dots, n/2$. It is easy to check out that this is really a decomposition into edge-disjoint Hamiltonian cycles.

6. If r is even n is odd then let us consider the Hamiltonian cycles H_1, \dots, H_{n-1} defined by (1), where t runs from 0 to $r-1$ and $i = 1, \dots, (n-1)/2$. The remaining edges form the set $H^* = \bigcup_{t=0}^{r-1} F_{t,0}$. The graph consisting of these edges is just $C_r + S_n$. In this manner the edges of H_1, \dots, H_{n-1}, H^* give out all the edges of $C_r \otimes S_n$.

3. MAIN RESULT

THEOREM. If G_1 and G_2 are decomposable into Hamiltonian cycles then their lexicographic product is decomposable, too.

If G_1 has r vertices and consists of p Hamiltonian cycles and if G_2 has n vertices and consists of k Hamiltonian cycles then $G_1 \otimes G_2$ is decomposable into $pn + k$ Hamiltonian cycles.

Proof. Counting the degrees it is clear that if the decomposition is possible, then the number of the Hamiltonian cycles is $pn + k$.

Since G_1 is the union of p graphs isomorphic to C_r , $G_1 \otimes G_2$ is the edge-disjoint union of $C_r \otimes G_2$ and $p-1$ graphs isomorphic to $C_r \otimes S_n$. As $C_r \otimes S_n$ is decomposable into Hamiltonian cycles (Section 2, 1-4) it is enough to prove that $C_r \otimes G_2$ is decomposable into $n + k$ Hamiltonian cycles. In the following we will prove this fact.

We remark that we can choose arbitrary the numbering of vertices in the class $V_t = \{x_{tj}, j = 0, \dots, n-1\}$. In other words the definition of lexicographic product can be formulated as follows:

The vertices of $G_1 \otimes G_2$ are partitioned in disjoint classes $V_t, |V_t| = n$ $t = 0, \dots, r-1$. The edges between the classes $V_t, V_{t'}$ are in $G_1 \otimes G_2$ if and only if (t, t') is an edge in G_1 , otherwise the edges among the vertices of V_t form a graph isomorphic to G_2 for every $t = 0, \dots, r-1$.

Let K_{ti} ($t = 0, \dots, r-1$, $i = 1, \dots, k$) be cycles of length n so that the vertex-set of K_{ti} is V_t and $\bigcup_{i=1}^k K_{ti}$ is isomorphic to G_2 . (Clearly $k \leq \lfloor (n-1)/2 \rfloor$.)

The base of our proof is that after a suitable numbering of the vertices we combine the edge-set $\bigcup_{t=0}^{r-1} K_{ti}$ with some Hamiltonian cycles of $C_r \otimes S_n$ and we decompose these edges into new Hamiltonian cycles.

Case A. r and n are odd.

Let us put the graph G_2 into V_t so that

(x_{t0}, x_{ti}) is an edge of K_{ti} if $t \in \{1, 3, \dots, r-4\} = T_1$,

$(x_{t,0}, x_{t,-i})$ is an edge of K_{ti} if $t \in \{0, 2, 4, \dots, r-5, r-3, r-2\} = T_2$,

$(x_{r-1,0}, x_{r-1,2i-1})$ is an edge of $K_{r-1,i}$.

This numbering is trivially possible: If $t \in T_1$ then we choose one point of G_2 for the role of 0. We give number i to one of the vertices which is connected with 0 in K_{ti} ($i = 1, \dots, k$) and we number the remaining vertices arbitrarily. Similarly we construct a numbering in the remaining classes V_t . In the case $t = r-1$ we utilize that $1 \leq i \leq k < n/2$, thus $0 \neq 2i-1$, and if $i_1 \neq i_2$ then $2i_1-1 \neq 2i_2-1$.

It is enough to show that $H_i \cup (\bigcup_{t=0}^{r-1} K_{ti})$ is decomposable into two Hamiltonian cycles, M_i^1 and M_i^2 ($i = 1, \dots, k$). (H_i is defined in 1.) See K_{t_2} ($t = 0, 1, \dots, 4$) in Fig. 2. Edges that do not occur in M_i^1 are drawn with a dashed line.

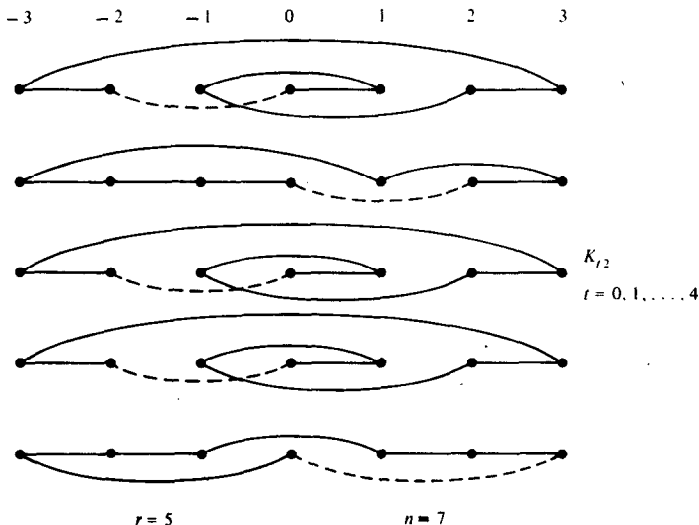


FIGURE 2

(a) Let

$$\begin{aligned}
 M_i^1 = & \bigcup_{t \in T_2} (K_{ti} \setminus (x_{t,-i}, x_{t,0}) \cup (x_{t,-i}, x_{t+1,0})) \\
 & \cup \bigcup_{t \in T_1} (K_{ti} \setminus (x_{t,i}, x_{t,0}) \cup (x_{t,i}, x_{t+1,0})) \\
 & \cup \bigcup (K_{r-1,i} \setminus (x_{r-1,2i-1}, x_{r-1,0}) \cup (x_{r-1,2i-1}, x_{0,0}))
 \end{aligned}$$

(see M_2^1 and M_2^2 in Fig. 3 for $r = 5$, $n = 7$ and a possible numbering of the vertices). Clearly M_i^1 is a Hamiltonian circuit.

(b) To prove that $M_i^2 = H_i \cup (\bigcup_{t=0}^{r-1} K_{ti}) \setminus M_i^1$ is also a Hamiltonian cycle we define $P_j (=P_j(i))$ as the sequence of the vertices:

$$\begin{aligned}
 P_j = & (x_{0,j}, x_{1,j+i}, x_{2,j}, x_{3,j+i}, \dots, \\
 & x_{r-4,j+i}, x_{r-3,j}, x_{r-2,j+i}, x_{r-1,j+2i}) \quad (j = 0, \dots, n-1).
 \end{aligned}$$

Clearly the consecutive members of P_j are connected in H_i . The last member of P_j is also connected with the first member of P_{j+1} .

It can be easily verified that M_i^2 strings the vertices of our graph to Hamilton cycle in the following order:

$$\begin{aligned}
 & x_{0,-i}, x_{0,0}, x_{1,i}, x_{1,0}, x_{2,-i}, x_{2,0} \dots x_{r-4,i}, \\
 & x_{r-4,0}, x_{r-3,-i}, x_{r-3,0}, x_{r-2,i}, x_{r-1,2i}, P_1, P_2 \dots \\
 & P_{n-2i-1}, (P_{n-2i} \setminus x_{r-1,0}), x_{r-2,0}, x_{r-1,i}, P_{-i+1}, \\
 & P_{-i+2}, \dots, P_{-1}, x_{r-1,0}, P_{-2i+1}, P_{-2i+2}, \dots, P_{-i-1}.
 \end{aligned}$$

Case B. r is odd, n is even.

Let us divide the class V_i into two parts, W_i and W'_i : $W_i = \{x_{ij}, j = 0, \dots, n/2 - 1\}$, $W'_i = \{x_{ij}, j = n/2, \dots, n-1\}$. In the decomposition of $C_r \otimes S_n$ (see 4) $n/2$ Hamiltonian cycles belong to the left cycle, $n/2$

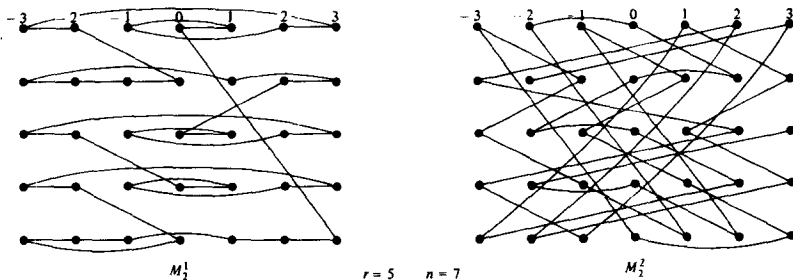


FIGURE 3

Hamiltonian cycles to the right cycle of Fig. 1. Now we will use the Hamiltonian cycles belonging to the left side of Fig. 1. These Hamiltonian cycles $H_0, \dots, H_{n/2-1}$ are defined in 2 as the decomposition of $C_{2r} \otimes S_{n/2}$.

Let us put the graph G_2 into V_t so that $(x_{t,n/2-i}, x_{t,n/2})$ is an edge of K_{ti} if $t = 0, 2, 4, \dots, r-1$. $(x_{t,0}, x_{t,n/2+i})$ is an edge of K_{ti} if $t = 1, 3, \dots, r-2$. As $1 \leq i \leq k \leq n/2-1$ the just-mentioned edges go between W_t and W'_t . This numbering can be made easily by a similar method described in Case A.

Now we decompose $H_i \cup (\bigcup_{t=0}^{r-1} K_{ti})$ into two Hamiltonian cycles M_i^1 and M_i^2 ($i = 1, 2, \dots, k$).

Let

$$\begin{aligned} M_i^1 = & \bigcup_{\substack{t \text{ even} \\ t \neq r-1}} (K_{ti} \setminus (x_{t,n/2-i}, x_{t,n/2}) \cup (x_{t,n/2-i}, x_{t+1,0})) \\ & \cup \bigcup_{t \text{ odd}} (K_{ti} \setminus (x_{t,0}, x_{t,n/2+i}) \cup (x_{t,n/2+i}, x_{t+1,n/2})) \\ & \cup \bigcup (K_{r-1,t} \setminus (x_{r-1,n/2-i}, x_{r-1,n/2}) \cup (x_{r-1,n/2-i}, x_{0,n/2})). \end{aligned}$$

Clearly, M_i^1 is a Hamiltonian cycle. Let $M_i^2 = (H_i \cup (\bigcup_t K_{ti})) \setminus M_i^1$. As in Case A(b) one can prove that this is a Hamiltonian cycle, too.

Case C. r and n are even.

In the same manner, what we have used in Case B we can decompose $H_i \cup (\bigcup_{t=0}^{r-1} K_{ti})$ into two Hamiltonian cycles M_i^1 and M_i^2 ($i = 1, \dots, k$).

Let

$$\begin{aligned} M_i^1 = & \bigcup_{t \text{ even}} (K_{ti} \setminus (x_{t,n/2-i}, x_{t,n/2}) \cup (x_{t,n/2-i}, x_{t+1,0})) \\ & \cup \bigcup_{t \text{ odd}} (K_{ti} \setminus (x_{t,0}, x_{t,n/2+i}) \cup (x_{t,n/2+i}, x_{t+1,n/2})), \\ M_i^2 = & \left(H_i \cup \left(\bigcup_t K_{ti} \right) \right) \setminus M_i^1. \end{aligned}$$

Case D. r is even, n is odd.

Now a construction similar to the preceding cases does not work: If we use the decomposition of 2 and an appropriate numbering of the vertices the graph M_i^2 is the union of two cycles in place of a Hamiltonian cycle. So we use the decomposition described in 6.

Let us put G_2 into V_t so that $(x_{t,0}, x_{t,2i-1})$ and $(x_{t,0}, x_{t,1-2i})$ are edges of K_{ti} $t = 0, \dots, r-1$, $i = 1, \dots, k$.

This can be reached easily: We choose one point of G_2 for the role of 0. We give numbers $2i-1$ and $1-2i$ to the vertices which are connected with

0 in K_{ti} . As n is odd and $k < n/2$ all these numbers are different mod n if i runs from 1 to k . We number the remaining vertices arbitrarily.

Now we could put G_2 into V_t in the same way for every t . It means that $H^* \cup (\bigcup_{t=0}^{r-1} K_{tk})$ is isomorphic to $C_r + C_n$. As we mentioned in the Introduction this graph is decomposable into two Hamiltonian circuits.

Now we decompose $H_{2i-1} \cup H_{2i} \cup (\bigcup_{t=0}^{r-1} K_{ti})$, $i = 1, \dots, k-1$ into three Hamiltonian cycles M_i^1, M_i^2, M_i^3 , which completes our proof.

$$M_i^1 = \bigcup_{t \text{ even}} (K_{ti} \setminus (x_{t,0}, x_{t,1-2i}) \cup (x_{t,1-2i}, x_{t+1,0})) \\ \cup \bigcup_{t \text{ odd}} (K_{ti} \setminus (x_{t,0}, x_{t,2i-1}) \cup (x_{t,2i-1}, x_{t+1,0})).$$

Let $P_j (=P_j(i))$, $R_j (=R_j(i))$, $Q_j (=Q_j(i))$ and $S_j (=S_j(i))$ be the following sequences of vertices:

$$P_j = (x_{0,j}, x_{1,j+2i}, x_{2,j+1}, x_{3,j+2i}, x_{4,j+1}, \dots, x_{r-1,j+2i}),$$

$$R_j = (x_{0,j}, x_{1,j+2i-1}, x_{2,j-1}, x_{3,j+2i-2}, \\ x_{4,j-1}, x_{5,j+2i-2}, \dots, x_{r-1,j+2i-2}),$$

$$Q_j = (x_{0,j}, x_{1,j+2i}, x_{2,j+1}, x_{3,j+2i+1}, \\ x_{4,j+1}, x_{5,j+2i+1}, \dots, x_{r-1,j+2i+1}),$$

$$S_j = (x_{0,j}, x_{r-1,j+2i}, x_{r-2,j}, x_{r-3,j+2i}, \dots, x_{1,j+2i}).$$

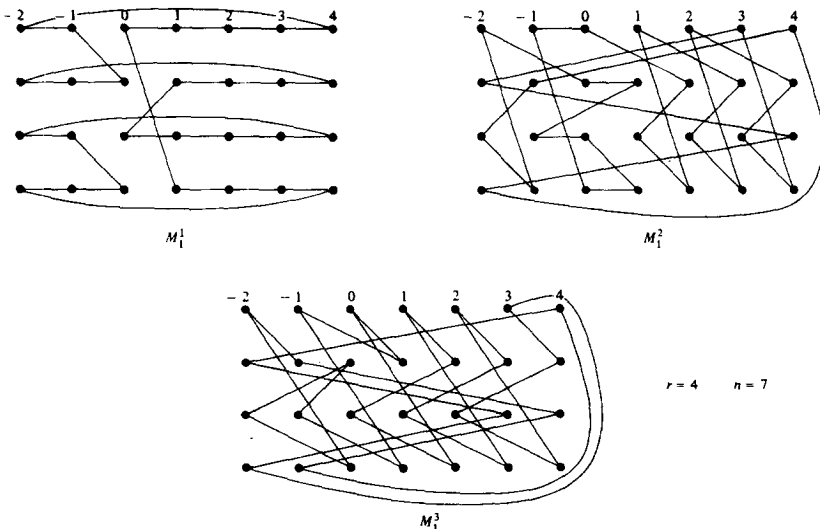


FIGURE 4

It can be seen that the consecutive members of P_j (resp. R_j , Q_j , S_j) are connected in $H_{2i-1} \cup H_{2i}$. The last member of P_j (resp. R_j , Q_j , S_j) is connected with the first element of P_{j+1} (resp. R_{j+1} , Q_{j+1} , S_{j+1}).

Let us define M_i^2 and M_i^3 by the following sequences (see Fig. 4 for $r = 4$, $n = 7$, $i = 1$).

$$\begin{aligned} M_i^2 = & (P_0, P_1, P_{n-2i-1}, x_{0,-2i}, \\ & x_{1,0}, (R_0 \setminus x_{0,0}), R_{n-1}, \\ & R_{n-2}, \dots, R_{n-2i+3}, x_{0,-2i+2}, x_{1,1}, x_{2,-2i+1}, \\ & x_{2,0}, x_{3,2i-1}, x_{3,0}, x_{4,1-2i}, x_{4,0}, \dots \\ & x_{r-1,2i-1}, x_{r-1,0}, x_{0,1-2i}) \end{aligned}$$

(in case $i = 1$ some elements of this sequence do not exist).

$$\begin{aligned} M_i^3 = & (Q_{n-2i+1}, Q_{n-2i+2}, \dots, Q_{n-2}, x_{0,-1}, x_{1,2i-1}, \\ & S_0, S_1, S_{n-2i-1}, (S_{n-2i} \setminus x_{1,0}), (Q_{n-2i} \setminus x_{0,n-2i})). \end{aligned}$$

The authors are indebted to the referee for suggesting that M_i^1 , M_i^2 , and M_i^3 are edge-disjoint Hamiltonian cycles. Q.E.D.

Remark 1. It would be possible to construct a decomposition in Case C with the method of Case D, too.

Remark 2. The fact that we find the same graph G_2 in the classes V_i was used only in Case D. It would be interesting to find a proof which does not use this.

Remark 3. If n and r are odd then the method of Case A is suitable to prove the following statement:

Let G_1 and G_2 two directed graphs in which there are no loops and multiple edges but where both (x, y) and (y, x) can occur as edges at the same time. Assume that G_1 and G_2 can be decompose into edge-disjoint directed Hamiltonian cycles. Then $G_1 \otimes G_2$ is also decomposable into directed Hamiltonian cycles.

We do not know whether this statement holds or not in the other cases.

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